UWThPh-6-1999

Graded Differential Geometry of Graded Matrix Algebras

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Abstract

We study the graded derivation-based noncommutative differential geometry of the $\mathbb{Z}_2$-graded algebra $\mathbb{M}(n|m)$ of complex $(n+m)\times(n+m)$-matrices with the "usual block matrix grading" (for $n \neq m$). Beside the (infinite-dimensional) algebra of graded forms the graded Cartan calculus, graded symplectic structure, graded vector bundles, graded connections and curvature are introduced and investigated. In particular we prove the universality of the graded derivation-based first-order differential calculus and show, that $\mathbb{M}(n|m)$ is a "noncommutative graded manifold" in a stricter sense: There is a natural body map and the cohomologies of $\mathbb{M}(n|m)$ and its body coincide (as in the case of ordinary graded manifolds).

1991 MSC:	17B56, 17B70, 46L87, 58A50, 58B30, 58C50, 81T60
PACS:	02.40.-k, 11.30.Pb
Keywords:	Supermanifolds, Lie superalgebras, noncommutative differential geometry, matrix geometry

1 Introduction

The basic idea of noncommutative geometry [2], that is the formulation of differential geometric concepts on more general algebras than the algebras of $\mathcal{C}^\infty$-functions on differentiable manifolds, is at least conceptionally rooted in the fact, that all the information about the

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[2]Research partly supported by the "Fonds zur Förderung der wissenschaftlichen Forschung (FWF)" through the research project P11783-PHY "Quantum Field Theory on Noncommutative Manifolds".

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differentiable manifold and its sheaf of differentiable functions is encoded in its algebra of global $\mathcal{C}^\infty$-functions such that differential geometry can be formulated in terms of the latter algebras. Although the $\mathbb{Z}_2$-graded algebra of global sections of the structure sheaf of a graded manifold is a "baby-noncommutative geometry" the differential geometry of graded manifolds is treated and interpreted in the spirit of classical differential and algebraic geometry. So graded manifolds should not be seen as specific noncommutative geometries to which the general methods of noncommutative geometry applies but rather as a conceptional starting point of a "super-generalization" of noncommutative geometry. Because graded manifolds are completely determined by the $\mathbb{Z}_2$-graded algebra of global sections of their structure sheafs [1] the natural class of objects to which such a generalization applies are $\mathbb{Z}_2$-graded real respectively complex algebras.

There exist already several articles and books in the literature dealing with various aspects of $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras, supersymmetry and noncommutative geometry. Without being complete let us just mention [20, 27], where notions as cyclic cohomology and Fredholm modules are treated in the $\mathbb{Z}_2$-graded setting, [10], where supersymmetry is employed to establish metric, Kähler and symplectic structures in noncommutative geometry, [19], where the concept of a spectral triple is extended to algebras which contain bosonic and fermionic degrees of freedom and [8, 21], where a possibility of generalizing matrix geometry to the $\mathbb{Z}_2$-graded framework is presented. Here we want to adopt a somewhat different point of view.

If $\mathcal{O}(\mathrm{X})$ is the $\mathbb{Z}_2$-graded algebra of global sections of the (complexified) structure sheaf of some graded manifold, (complex) global graded vector fields on the graded manifold are by definition graded derivations of $\mathcal{O}(\mathrm{X})$. All global graded vector fields $\mathfrak{V}^g(\mathrm{X})$ form a complex Lie subsuperalgebra and a graded module over the graded center $\mathcal{Z}^g(\mathcal{O}(\mathrm{X}))$. (Complex) global graded p-forms for $p \in \mathbb{N}$ are defined as p-fold $\mathcal{Z}^g(\mathcal{O}(\mathrm{X}))$-graded-multilinear, graded-alternating maps from $\mathfrak{V}^g(\mathrm{X})$ to $\mathcal{O}(\mathrm{X})$ and one can form the $\mathbb{N}_0 \times \mathbb{Z}_2$-bigraded $\mathbb{C}$-vector space $\Omega^g(\mathrm{X})$ of global graded forms as direct sum of all $\mathbb{Z}_2$-graded $\mathbb{C}$-vector spaces $\Omega^{g,p}(\mathrm{X})$ of global graded p-forms. The graded wedge product as well as the whole graded Cartan calculus on $\Omega^g(\mathrm{X})$ can be introduced (see [1, 22] for example) by employing only the facts, that $\mathfrak{V}^g(\mathrm{X})$ is a $\mathbb{C}$-Lie superalgebra and $\mathbb{Z}_2$-graded $\mathcal{Z}^g(\mathcal{O}(\mathrm{X}))$-module and that $\mathcal{O}(\mathrm{X})$ is a $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra. The important feature of the recipe for the construction of the graded deRham complex and the graded Cartan calculus formulated above is, that it uses only the graded algebra structure of $\mathcal{O}(\mathrm{X})$. In particular it does not use the graded commutativity of $\mathcal{O}(\mathrm{X})$ and we can define on arbitrary $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras noncommutative graded differential calculi.

What we have just described is mutatis mutandis the basic idea of the so-called derivation-based differential calculi [5, 6, 9] transposed to the $\mathbb{Z}_2$-graded setting. Such graded derivation-based differential calculi were investigated for arbitrary, but graded-commutative $\mathbb{Z}_2$-graded algebras in the framework of $\mathbb{Z}_2$-graded Lie-Cartan pairs [13, 20, 27].

Motivated by the rich differential geometric structure of ordinary matrix algebras [5, 6, 24] and by the our previous investigation of the fuzzy supersphere [15, 16], where each truncated supersphere was a graded matrix algebra in particular, we will investigate especially the differential calculus based on all graded derivations on the $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra $\mathbb{M}(n|m)$ of complex

$(n+m)\times(n+m)$-matrices $(n,m\in\mathbb{N}_0, n+m\in\mathbb{N})$, $\mathbb{Z}_2$-graded by declaring the vector subspace

$$\mathbb{M}(n|m)_{\overline{0}} := \left\{ M = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix} \,\Big|\, M_1 \in \mathbb{M}(n), M_4 \in \mathbb{M}(m) \right\} \tag{1}$$

of $\mathbb{M}(n|m)$ as even and the vector subspace

$$\mathbb{M}(n|m)_{\overline{1}} := \left\{ M = \begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix} \,\Big|\, M_2 \in \mathbb{M}(n,m), M_3 \in \mathbb{M}(m,n) \right\} \tag{2}$$

of $\mathbb{M}(n|m)$ as odd. Here $\mathbb{M}(n,m)$ and $\mathbb{M}(n)$ denote the vector space of $n\times m$- respectively the algebra of $n\times n$-matrices and we will always assume $n \neq m$.

Chapter 2 is devoted to the precise definition of the graded derivation-based differential calculus on $\mathbb{M}(n|m)$ as described above and its immediate consequences. The resulting complexes are nothing else than the complexes of Lie superalgebra cohomology with values in $\mathbb{M}(n|m)$ and typically infinite. The latter fact shows in particular, that the complex is completely different to that proposed in [8, 21].

In chapter 3 we continue the investigation of the differential calculus using the facts, that there exist graded-commutative homogeneous bases in the $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-modules of all graded p-forms and that all graded derivations of $\mathbb{M}(n|m)$ are inner. Especially we construct an invariant graded 1-form, which determines the differential of graded matrices in terms of graded commutators (within the graded algebra of graded forms) and show, that the first-order differential calculus is universal.

Associated with every graded manifold there exists an even, surjective algebra homomorphism β_{X} from the $\mathbb{Z}_2$-graded algebra of global sections of the (complexified) structure sheaf of the graded manifold $\mathcal{O}(\mathrm{X})$ to the algebra $\mathcal{C}^\infty(\mathrm{X})$ of (complex) $\mathcal{C}^\infty$-functions on its body manifold X. We call this map, which is the key to all further developments in graded manifold theory, the body map. In chapter 4 we show, that there exists a natural noncommutative analogue to the body map. It induces an isomorphism between the graded derivation-based cohomology of $\mathbb{M}(n|m)$ and the derivation-based cohomology of its body, such that the situation described by a theorem of Kostant [22] is generalized to the noncommutative case.

In chapter 5 we study the graded symplectic geometry of $\mathbb{M}(n|m)$. As for ordinary matrix geometry [5, 6], which is included as special case, there exists a graded symplectic structure such that the induced graded Poisson bracket on $\mathbb{M}(n|m)$ is (i times) the graded commutator on $\mathbb{M}(n|m)$.

In the last chapter we investigate the noncommutative generalization of graded vector bundles over graded manifolds. Graded vector bundles over a graded manifold $(\mathrm{X},\mathcal{O})$ are usually introduced as locally graded-free $\mathcal{O}$-modules [1, 17, 22]. In the spirit of noncommutative geometry [2, 14] we concentrate on the module of global sections and introduce graded vector bundles over $\mathbb{M}(n|m)$ as $\mathbb{Z}_2$-graded, finitely generated (graded-projective) $\mathbb{M}(n|m)$-modules. Concepts like connections and curvature can be generalized to the $\mathbb{Z}_2$-graded noncommutative setting.

In addition we have included an appendix in which we analyze the associative product of supertrace-free, graded matrices. The results of this analysis are used for a minimality proof in chapter 3.

There will appear lots of $\mathbb{Z}_2$-graded objects. If the object is denoted by $\mathcal{A}$ its even part is denoted by $\mathcal{A}_{\overline{0}}$, its odd part by $\mathcal{A}_{\overline{1}}$. If a is some homogeneous element of such an object its degree will be denoted by $\overline{a}$. Speaking of grading in the context of an ungraded object we mean, that the object is endowed with its trivial graduation. If for some construction the $\mathbb{Z}_2$-grading is indicated by an index "g" we omit this index in the case of trivial graduation.

2 Graded derivation-based differential calculus on graded matrix algebras

We will interpret the $\mathbb{C}$-Lie superalgebra and $\mathbb{Z}_2$-graded $\mathcal{Z}^g(\mathbb{M}(n|m))$-module $\mathfrak{Der}^g(\mathbb{M}(n|m))$ of all graded derivations of $\mathbb{M}(n|m)$ as "'noncommutative graded vector fields" on $\mathbb{M}(n|m)$. Because $\mathbb{M}(n|m)$ is graded-central,

$$\mathcal{Z}^g(\mathbb{M}(n|m)) = \mathcal{Z}^g(\mathbb{M}(n|m))_{\overline{0}} = \mathbb{C}\,1_{n+m} \cong \mathbb{C}\,, \tag{3}$$

the concept of graded $\mathcal{Z}^g(\mathbb{M}(n|m))$-multilinearity reduces to ordinary $\mathbb{C}$-multilinearity and we can employ ideas and results of Lie superalgebra cohomology (see [11, 28]) for the construction of the graded derivation-based differential calculus on $\mathbb{M}(n|m)$.

For every natural number $p \in \mathbb{N}$ let us denote by $\mathrm{Hom}^p(\mathfrak{Der}^g(\mathbb{M}(n|m)); \mathbb{M}(n|m))$ the $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space of all p-linear maps $\mathfrak{Der}^g(\mathbb{M}(n|m)) \times \overset{p}{\cdots} \times \mathfrak{Der}^g(\mathbb{M}(n|m)) \longrightarrow \mathbb{M}(n|m)$ and by $\mathfrak{S}_p$ the symmetric group of p letters. Introducing a commutation factor $\gamma_p : \mathfrak{S}_p \times \mathbb{Z}_2 \times \overset{p}{\cdots} \times \mathbb{Z}_2 \longrightarrow \{\pm 1\}$ via

$$\gamma_p(\sigma; \overline{i}_1, \cdots, \overline{i}_p) := \prod_{\substack{r,s=1,\cdots,p; r<s \\ \sigma^{-1}(r) > \sigma^{-1}(s)}} (-1)^{\overline{i}_r \overline{i}_s}\,, \tag{4}$$

we can define a representation π of $\mathfrak{S}_p$ on $\mathrm{Hom}^p(\mathfrak{Der}^g(\mathbb{M}(n|m)); \mathbb{M}(n|m))$ by

$$(\pi_\sigma \omega)(D_1, \cdots, D_p) := \gamma_p(\sigma; \overline{D}_1, \cdots, \overline{D}_p)\omega(D_{\sigma(1)}, \cdots, D_{\sigma(p)}) \tag{5}$$

for all $\omega \in \mathrm{Hom}^p(\mathfrak{Der}^g(\mathbb{M}(n|m)); \mathbb{M}(n|m))$, all homogeneous $D_1, \cdots, D_p \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ and all $\sigma \in \mathfrak{S}_p$. Now by definition a p-linear map $\omega \in \mathrm{Hom}^p(\mathfrak{Der}^g(\mathbb{M}(n|m)); \mathbb{M}(n|m))$ is called graded-alternating if

$$\pi_\sigma \omega = \mathrm{sgn}\sigma\, \omega \tag{6}$$

is fulfilled for all $\sigma \in \mathfrak{S}_p$ and we interpret such maps as graded p-forms on $\mathbb{M}(n|m)$. All graded p-forms on $\mathbb{M}(n|m)$ form a graded vector subspace of $\mathrm{Hom}^p(\mathfrak{Der}^g(\mathbb{M}(n|m)); \mathbb{M}(n|m))$, which we will denote by $\Omega^{g,p}(\mathbb{M}(n|m))$.
A general graded form on $\mathbb{M}(n|m)$ is an element of the direct sum

$$\Omega^g(\mathbb{M}(n|m)) := \bigoplus_{p \in \mathbb{N}_0} \Omega^{g,p}(\mathbb{M}(n|m))\,, \tag{7}$$

where we set $\Omega^{g,0}(\mathbb{M}(n|m)) := \mathbb{M}(n|m)$. Employing the multiplicative structure of $\mathbb{M}(n|m)$ we can proceed exactly as in the case of graded manifolds [1, 22] (respectively graded Lie-Cartan pairs [13, 27]) to introduce a graded wedge product on $\Omega^g(\mathbb{M}(n|m))$. So we first define for all

$p, p' \in \mathbb{N}_0, \overline{i}, \overline{i'} \in \mathbb{Z}_2$ a bilinear map $\wedge : \Omega^{g,p}(\mathbb{M}(n|m))_{\overline{i}} \times \Omega^{g,p}(\mathbb{M}(n|m))_{\overline{i'}} \longrightarrow \Omega^{g,p+p'}(\mathbb{M}(n|m))_{\overline{i+i'}}$ by

$$\begin{aligned}(\omega \wedge \omega')(D_1, \cdots, D_{p+p'}) := \frac{1}{p!p'!} \sum_{\sigma \in \mathfrak{S}_{p+p'}} \mathrm{sgn}\sigma\, \gamma_{p+p'}(\sigma; \overline{D}_1, \cdots, \overline{D}_{p+p'}) \cdot \\ \cdot (-1)^{\overline{i'} \sum_{l=1}^{p} \overline{D}_{\sigma(l)}} \omega(D_{\sigma(1)}, \cdots, D_{\sigma(p)}) \omega'(D_{\sigma(p+1)}, \cdots, D_{\sigma(p+p')})\end{aligned} \tag{8}$$

for all homogeneous $D_1, \cdots, D_{p+p'} \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ and extend these by bilinearity to $\Omega^g(\mathbb{M}(n|m))$. With respect to it $\Omega^g(\mathbb{M}(n|m))$ becomes a $\mathbb{N}_0 \times \mathbb{Z}_2$-bigraded $\mathbb{C}$-algebra.

Via

$$\begin{aligned}(L_{D_0}\omega)(D_1, \cdots, D_p) \;:=\;& D_0(\omega(D_1, \cdots, D_p)) - \\ &- \sum_{l=1}^{p} (-1)^{\overline{D}_0(\overline{\omega} + \sum_{l'=1}^{l-1} \overline{D}_{l'})} \omega(D_1, \cdots, [D_0, D_l]_g, \cdots, D_p),\end{aligned} \tag{9}$$

$$(\imath_{D_1}\omega)(D_2, \cdots, D_p) := \omega(D_1, D_2, \cdots, D_p) \tag{10}$$

and

$$\begin{aligned}d\omega(D_0, \cdots, D_p) = \sum_{l=0}^{p} (-1)^{l + \overline{D}_l(\overline{\omega} + \sum_{l'=0}^{l-1} \overline{D}_{l'})} L_{D_l}\left(\omega(D_0, \cdots, \overset{\vee}{D}_l, \cdots, D_p)\right) + \\ + \sum_{0 \le l < l' \le p} (-1)^{l' + \overline{D}_{l'} \sum_{l''=l+1}^{l'-1} \overline{D}_{l''}} \omega(D_0, \cdots, D_{l-1}, [D_l, D_{l'}]_g, \cdots, \overset{\vee}{D}_{l'}, \cdots, D_p)\end{aligned} \tag{11}$$

for all homogeneous $D_0, D_1, \cdots, D_p \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ and all homogeneous $\omega \in \Omega^{g,p}(\mathbb{M}(n|m))$ ($\vee$ denotes omission), one defines homogeneous endomorphisms $\Omega^g(\mathbb{M}(n|m)) \longrightarrow \Omega^g(\mathbb{M}(n|m))$ of bidegree $(0, \overline{D}_0)$, $(-1, \overline{D}_0)$ and $(1, \overline{0})$ respectively. The assignments $D \mapsto \imath_D$ and $D \mapsto L_D$ extend to $\mathbb{C}$-linear maps $\mathfrak{Der}^g(\mathbb{M}(n|m)) \longrightarrow \mathrm{End}(\Omega^g(\mathbb{M}(n|m)))$ and L is a graded representation of $\mathfrak{Der}^g(\mathbb{M}(n|m))$ in particular. Furthermore the relations

$$\begin{aligned} d \circ d &= 0 \\ d \circ L_D &= L_D \circ d \end{aligned} \tag{12}$$

as well as

$$\begin{aligned} \imath_D \circ \imath_{D'} + (-1)^{\overline{DD'}} \imath_{D'} \circ \imath_D &= 0 \\ (L_D \circ \imath_{D'} - \imath_{D'} \circ L_D)\,\omega &= (-1)^{\overline{D}\overline{\omega}} \imath_{[D,D']_g} \omega \\ (\imath_D \circ d + d \circ \imath_D)\,\omega &= (-1)^{\overline{D}\overline{\omega}} L_D \omega \end{aligned} \tag{13}$$

for all homogeneous $D, D' \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ and all bihomogeneous $\omega \in \Omega^g(\mathbb{M}(n|m))$ are known from Lie superalgebra cohomology [28].

By analogy with the case of graded manifolds we call d, L_D and $\imath_D$ exterior derivative, Lie derivative and interior product (with respect to a graded vector field $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))$). (12) and (13) tell us, that they fulfill exactly the same relations as in the "graded-commutative case", but this observation remains also true for the graded wedge product (8) of graded forms.

Proposition 1 *The relations*

$$
\begin{aligned}
L_D\left(\omega \wedge \omega'\right) &= \left(L_D \omega\right) \wedge \omega' + (-1)^{\overline{D}\overline{\omega}} \omega \wedge L_D \omega' \\
\imath_D\left(\omega \wedge \omega'\right) &= (-1)^{\overline{D}\overline{\omega'}}\left(\imath_D \omega\right) \wedge \omega' + (-1)^p \omega \wedge \imath_D \omega' \\
d\left(\omega \wedge \omega'\right) &= (d\omega) \wedge \omega' + (-1)^p \omega \wedge d\omega'
\end{aligned}
\tag{14}
$$

are fulfilled for all homogeneous $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))$, $\omega \in \Omega^{g,p}(\mathbb{M}(n|m))$, $\omega' \in \Omega^{g,p'}(\mathbb{M}(n|m))$.

Proof: This can be shown exactly as in the case of graded manifolds. That is, one starts with a direct proof of the second relation and proofs the other equations inductively using the last two relations (13). □

Because we interpret d as exterior derivative, the Lie superalgebra cohomology of $\mathfrak{Der}^g(\mathbb{M}(n|m))$ with values in $\mathbb{M}(n|m)$,

$$
H(\mathbb{M}(n|m)) \equiv \bigoplus_{p \in \mathbb{N}_0} H^p(\mathbb{M}(n|m)) := \frac{\ker d}{\operatorname{im} d}\,, \tag{15}
$$

has to be seen as analogue to the graded deRham-cohomology on graded manifolds. Via

$$
[\omega] \wedge [\omega'] := [\omega \wedge \omega'] \tag{16}
$$

the above graded derivation-based cohomology of $\mathbb{M}(n|m)$ becomes a $\mathbb{N}_0 \times \mathbb{Z}_2$-bigraded $\mathbb{C}$-algebra and we will continue to study it in chapter 4.

3 Homogeneous bases and the canonical graded 1-form

Whereas the definitions and results of the preceding considerations apply to each $\mathbb{Z}_2$-graded, graded-central $\mathbb{C}$-algebra we shall now employ more specific properties of $\mathbb{M}(n|m)$. There will result similar formulas as in "ordinary" matrix geometry [5, 6, 24], which is included as special case.

The sets $\Omega^{g,p}_{\mathcal{Z}^g}(\mathbb{M}(n|m))$ of graded p-forms with values in the graded center of $\mathbb{M}(n|m)$ form graded vector subspaces of $\Omega^{g,p}(\mathbb{M}(n|m))$ for all $p \in \mathbb{N}$ and one can introduce

$$
\Omega^{g}_{\mathcal{Z}^g}(\mathbb{M}(n|m)) := \bigoplus_{p \in \mathbb{N}_0} \Omega^{g,p}_{\mathcal{Z}^g}(\mathbb{M}(n|m)) \tag{17}
$$

with $\Omega^{g,0}_{\mathcal{Z}^g}(\mathbb{M}(n|m)) = \mathcal{Z}^g(\mathbb{M}(n|m))$. $\Omega^{g}_{\mathcal{Z}^g}(\mathbb{M}(n|m))$ is a bigraded subalgebra of $\Omega^g(\mathbb{M}(n|m))$, whose product fulfills

$$
\omega \wedge \omega' = (-1)^{pp' + \overline{\omega}\overline{\omega}'} \omega' \wedge \omega \tag{18}
$$

for all homogeneous $\omega \in \Omega^{g,p}_{\mathcal{Z}^g}(\mathbb{M}(n|m)), \omega' \in \Omega^{g,p'}_{\mathcal{Z}^g}(\mathbb{M}(n|m))$ and which is stable with respect to the whole Cartan calculus.

Now let us introduce a homogeneous basis $\{\partial_A\}_{A=1,\cdots,n'+m'}$ of $\mathfrak{Der}^g(\mathbb{M}(n|m))$ with $\partial_1, \cdots, \partial_{n'} \in \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{0}}$, $\partial_{n'+1}, \cdots, \partial_{n'+m'} \in \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{1}}$, where we set $n' := \dim_{\mathbb{C}} \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{0}}$

and $m' := \dim_{\mathbb{C}} \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{1}}$. If $\{\eta^A\}_{A=1,\cdots,n'+m'}$ denotes the dual basis to $\{\partial_A\}_{A=1,\cdots,n'+m'}$ we can introduce a homogeneous basis $\{\theta^A\}_{A=1,\cdots,n'+m'}$ of $\Omega^{g,1}_{\mathcal{Z}^g}(\mathbb{M}(n|m))$ by

$$\theta^A(D) := \eta^A(D) 1_{n+m} \tag{19}$$

for all $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))$. Employing the standard isomorphisms between graded-alternating maps and the graded exterior algebra [1, 28] one deduces, that

$$\left\{ \theta^{A_1} \wedge \cdots \wedge \theta^{A_p} \;\middle|\; (A_1, \cdots, A_p) \in \mathfrak{I}^{n'|m'}_p \right\} \tag{20}$$

with

$$\begin{aligned} \mathfrak{I}^{n'|m'}_p := \Big\{ & (k_1, \cdots, k_{p'}, \alpha_{p'+1}, \cdots, \alpha_p) \;\Big|\; 0 \le p' \le p; k_1, \cdots, k_{p'} = 1, \cdots, n'; \\ & \alpha_{p'+1}, \cdots, \alpha_p = n'+1, \cdots, n'+m'; k_1 < k_2 < \cdots < k_{p'} < \alpha_{p'+1} \le \cdots \le \alpha_{p-1} \le \alpha_p \Big\} \end{aligned} \tag{21}$$

is a homogeneous basis of $\Omega^{g,p}_{\mathcal{Z}^g}(\mathbb{M}(n|m)), p \in \mathbb{N}$.
Because of

$$M \wedge \omega = (-1)^{\overline{M}\overline{\omega}} \omega \wedge M \tag{22}$$

for all homogeneous $M \in \mathbb{M}(n|m)$ and all bihomogeneous $\omega \in \Omega^g_{\mathcal{Z}^g}(\mathbb{M}(n|m))$, the $\mathbb{N}_0 \times \mathbb{Z}_2$-bigraded $\mathbb{C}$-algebras $\Omega^g(\mathbb{M}(n|m))$ and $\mathbb{M}(n|m) \hat{\otimes}_{\mathbb{C}} \Omega^g_{\mathcal{Z}^g}(\mathbb{M}(n|m))$, where $\hat{\otimes}$ denotes the tensor product of $\mathbb{Z}_2$-graded algebras, are canonically isomorphic. In particular we can conclude:

Proposition 2 *The $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-bimodules $\Omega^{g,p}(\mathbb{M}(n|m))$ are graded-free for both multiplications and for all $p \in \mathbb{N}_0$. The set (20) determines a homogeneous basis of the left (right), $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-module $\Omega^{g,p}(\mathbb{M}(n|m))$.* □

Consequently every $\omega \in \Omega^{g,p}(\mathbb{M}(n|m))$ can be written as

$$\omega = \sum_{(A_1, \cdots, A_p) \in \mathfrak{I}^{n'|m'}_p} \omega_{A_1 \cdots A_p} \wedge \theta^{A_1} \wedge \cdots \wedge \theta^{A_p} \tag{23}$$

with unique coefficients $\omega_{A_1 \cdots A_p} \in \mathbb{M}(n|m)$. Explicitely these coefficients are given by

$$\omega_{A_1 \cdots A_p} = (-1)^{\frac{1}{2} p''(p''-1)} \frac{1}{\prod_{l=1}^{n'+m'} N_l!} \, \omega(\partial_{A_1}, \cdots, \partial_{A_p}) \,, \tag{24}$$

where p'' is the number of entries in $(A_1, \cdots, A_p)$ greater than n' and N_l is the number of entries in $(A_1, \cdots, A_p)$ being equal l.

In order to investigate graded derivations of $\mathbb{M}(n|m)$ (we include the case $n = m$ for the moment) let us denote by $\mathfrak{gl}(n|m)$ the (complex) general linear Lie superalgebra and by $\mathfrak{sl}(n|m)$ the (complex) special linear Lie superalgebra. The adjoint representation of $\mathfrak{gl}(n|m)$ is at the same time a Lie superalgebra homomorphism ad : $\mathfrak{gl}(n|m) \longrightarrow \mathfrak{Der}^g(\mathbb{M}(n|m))$ and, as we will see, the structure of $\mathfrak{Der}^g(\mathbb{M}(n|m))$ and its Lie subsuperalgebras is determined by this homomorphism.

Proposition 3 *If $\mathfrak{L}$ is a Lie subsuperalgebra of $\mathfrak{gl}(n|m)$ then*

$$\mathfrak{L}^{\mathrm{ad}} := \mathrm{im}\, \mathrm{ad}|_{\mathfrak{L}} \tag{25}$$

is a Lie subsuperalgebra of $\mathfrak{Der}^g(\mathbb{M}(n|m))$. Contrary every Lie subsuperalgebra of $\mathfrak{Der}^g(\mathbb{M}(n|m))$ is of this form. There are two different cases:

(i.) *For $n \neq m$ the restriction of* ad *to $\mathfrak{sl}(n|m)$ is an Lie superalgebra isomorphism onto $\mathfrak{Der}^g(\mathbb{M}(n|m))$ and the various restrictions of* ad *induce a bijective correspondence between Lie subsuperalgebras of $\mathfrak{sl}(n|m)$ and Lie subsuperalgebras of $\mathfrak{Der}^g(\mathbb{M}(n|m))$.*

(ii.) *For $n = m$ there is no Lie subsuperalgebra $\mathfrak{L}$ of $\mathfrak{gl}(n|n)$ such that the restriction of* ad *to $\mathfrak{L}$ becomes an Lie superalgebra isomorphism onto $\mathfrak{Der}^g(\mathbb{M}(n|n))$.*

Proof: An even graded derivation of $\mathbb{M}(n|m)$ is just an ordinary derivation of the $\mathbb{C}$-algebra $\mathbb{M}(n+m)$ and these are inner, because the first Hochschild cohomology group of $\mathbb{M}(n+m)$ with values in $\mathbb{M}(n+m)$ is trivial [26]. Introducing

$$\Gamma := \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix}$$

we find for some $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))_{\bar{1}}$ and all homogeneous $M \in \mathbb{M}(n|m)$

$$DM = \mathrm{ad}\left(\frac{1}{2}(D\Gamma)\Gamma\right)(M)\,,$$

from which we can conclude, that D is inner. Consequently, if $\mathfrak{D}$ is a Lie subsuperalgebra of $\mathfrak{Der}^g(\mathbb{M}(n|m))$, then $\mathfrak{L} := \mathrm{ad}^{-1}(\mathfrak{D})$ is a Lie subsuperalgebra of $\mathfrak{gl}(n|m)$ with $\mathfrak{L}^{\mathrm{ad}} = \mathfrak{D}$. (i.) and (ii.) are consequences of $1_{n+m} \notin \mathfrak{sl}(n|m)$ for $n \neq m$ respectively $1_{2n} \in \left[\mathfrak{gl}(n|n)_{\bar{1}}, \mathfrak{gl}(n|n)_{\bar{1}}\right]_g$. $\square$

The ultimate reason for restricting our geometric investigation to the case $n \neq m$ lies in the existence of the Lie superalgebra isomorphism $\mathrm{ad} : \mathfrak{sl}(n|m) \longrightarrow \mathfrak{Der}^g(\mathbb{M}(n|m))$. The elements of every homogeneous basis $\{\partial_A\}_{A=1,\cdots,n'+m'}$ of $\mathfrak{Der}^g(\mathbb{M}(n|m))$ are given by

$$\partial_A = \mathrm{ad}\, E_A\,, \tag{26}$$

where $\{E_A\}_{A=1,\cdots,n'+m'}$ is a homogeneous basis of $\mathfrak{sl}(n|m)$ and we have $n' = n^2+m^2-1, m' = 2nm$ in particular. Moreover, the structure constants c_{AB}^{C} appearing in

$$[\partial_A, \partial_B]_g = \sum_{C=1}^{(n+m)^2-1} c_{AB}^{C}\partial_C \tag{27}$$

are the structure constants of the homogeneous $\mathfrak{sl}(n|m)$-basis $\{E_A\}_{A=1,\cdots,(n+m)^2-1}$ and one deduces the nice formulas

$$dE_A = -\sum_{B,C=1}^{(n+m)^2-1} c_{AB}^{C} E_C \wedge \theta^B \tag{28}$$

and

$$d\theta^A = \frac{1}{2}\sum_{B,C=1}^{(n+m)^2-1} c_{BC}^{A}\theta^C \wedge \theta^B\,. \tag{29}$$

The even graded 1-form

$$\Theta := \sum_{A=1}^{(n+m)^2-1} E_A \wedge \theta^A \tag{30}$$

will be called canonical graded 1-form, because it plays a distinguished role.

Proposition 4 *The definition of Θ is independent of the choice of the homogeneous basis of $\mathfrak{Der}^g(\mathbb{M}(n|m))$. Θ is ($\mathfrak{Der}^g(\mathbb{M}(n|m))$-)invariant and this property determines Θ up to constant multiples. Furthermore its exterior differential fulfills*

$$d\Theta = \Theta \wedge \Theta \tag{31}$$

and the exterior differential of each $M \in \mathbb{M}(n|m)$ can be expressed according to

$$dM = [\Theta, M]_g \equiv \Theta \wedge M - M \wedge \Theta\,. \tag{32}$$

Proof : Beside the uniqueness statement only simple calculations are involved (for which one can use (28) and (29) advantageously). The irreducibility of the adjoint representation of $\mathfrak{sl}(n|m)$ [4] guarantees, that

$$L_D \omega = 0\,, \quad \omega \in \Omega^{g,1}(\mathbb{M}(n|m))\,,$$

for all $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ implies $\omega = c\Theta, c \in \mathbb{C}$. □

Finally we note, that $\Omega^g(\mathbb{M}(n|m))$ is in a certain sense minimal.

Proposition 5 *(28) can be inverted according to*

$$\theta^A = 4(n-m)^2 \sum_{B,C,D=1}^{(n+m)^2-1} (-1)^{\overline{E}_B \overline{E}_D} K^{AB} K^{CD} E_C E_B \wedge dE_D\,, \tag{33}$$

where K is the Killing form of $\mathfrak{sl}(n|m)$ and K^{AB} denote the components of the inverse matrix of $(K(E_A, E_B))$. Consequently, if Ω is differential subalgebra of $\Omega^g(\mathbb{M}(n|m))$ containing $\mathbb{M}(n|m)$, then $\Omega = \Omega^g(\mathbb{M}(n|m))$.

Proof : The minimality statement follows from (33) because of proposition 2. In order to show (33) one uses (28) and expands the threefold product of the basis elements E_A according to (A4). Using the results of proposition A (33) follows. □

The second part of proposition 5 can be stated differently: The canonical even algebra homomorphisms from the (intrinsic) $\mathbb{Z}_2$-graded universal differential envelope of $\mathbb{M}(n|m)$ to $\Omega^g(\mathbb{M}(n|m))$ (see [3, 20] for a precise definition) is onto. The restriction of this homomorphism to the corresponding first-order differential calculi is an isomorphism.

4 Cohomology and the noncommutative body map

We will call the even, surjective $\mathbb{C}$-linear map

$$\beta : \mathbb{M}(n|m) \longrightarrow \mathbb{M}(\underline{n}) \qquad \text{with} \quad \underline{n} := \begin{cases} n & \text{if } n > m \\ m & \text{if } n < m \end{cases} \tag{34}$$

defined by

$$\beta(M) \equiv \beta\left(\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\right) := \begin{cases} M_1 & \text{if } \underline{n} = n \\ M_4 & \text{if } \underline{n} = m \end{cases} \tag{35}$$

canonical body map of $\mathbb{M}(n|m)$. A justification for choosing this terminology will result from the investigation of its properties: They are completely analogous to the corresponding map of graded manifolds if one takes the noncommutativity of $\mathbb{M}(n|m)$ and its "body" $\mathbb{M}(\underline{n})$ appropriately into account. In order to distinguish between "objects" on $\mathbb{M}(n|m)$ and corresponding "objects" on the body we underline the latter.

The restriction of β to $\mathcal{Z}^g(\mathbb{M}(n|m))$ is an even algebra homomorphism onto $\mathcal{Z}(\mathbb{M}(\underline{n}))$ and by

$$\iota(\underline{M}) := \begin{cases} \begin{pmatrix} \underline{M} & 0 \\ 0 & \frac{1}{n}\mathrm{Tr}(\underline{M})1_m \end{pmatrix} & \text{if } \underline{n} = n \\ \\ \begin{pmatrix} \frac{1}{m}\mathrm{Tr}(\underline{M})1_n & 0 \\ 0 & \underline{M} \end{pmatrix} & \text{if } \underline{n} = m \end{cases}, \tag{36}$$

we can introduce an even, injective $\mathbb{C}$-linear map $\iota : \mathbb{M}(\underline{n}) \longrightarrow \mathbb{M}(n|m)$, which is right-inverse to β on the one hand and whose restriction to $\mathcal{Z}(\mathbb{M}(\underline{n}))$ is an even algebra homomorphism into $\mathcal{Z}^g(\mathbb{M}(n|m))$ on the other hand.
Analogous to the body map of graded manifolds β induces a Lie algebra homomorphism $\hat{\beta} : \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{0}} \longrightarrow \mathfrak{Der}(\mathbb{M}(\underline{n}))$ via

$$\hat{\beta}(D)\beta(M) := \beta(DM) \tag{37}$$

for all $M \in \mathbb{M}(n|m)$. $\hat{\beta}$ is surjective because of

$$\hat{\beta}(\mathrm{ad}\, E) = \mathrm{ad}\, \beta(E) \tag{38}$$

for all $E \in \mathfrak{sl}(n|m)_{\overline{0}}$ and in addition $\hat{\iota} : \mathfrak{Der}(\mathbb{M}(\underline{n})) \longrightarrow \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{0}}$,

$$\hat{\iota}(\mathrm{ad}\, \underline{E}) := \mathrm{ad}\, \iota(\underline{E}) \tag{39}$$

is a Lie algebra homomorphism right-inverse to $\hat{\beta}$.
Now we can introduce even $\mathbb{C}$-linear maps $\beta^{(p)} : \Omega^{g,p}(\mathbb{M}(n|m)) \longrightarrow \Omega^p(\mathbb{M}(\underline{n})), p \in \mathbb{N}$, by

$$\left(\beta^{(p)}(\omega)\right)\left(\underline{D}_1, \cdots, \underline{D}_p\right) := \beta\left(\omega\left(\hat{\iota}(\underline{D}_1), \cdots, \hat{\iota}(\underline{D}_p)\right)\right) \tag{40}$$

for all $\underline{D}_1, \cdots, \underline{D}_p \in \mathfrak{Der}(\mathbb{M}(\underline{n}))$. $\mathfrak{sl}(n|m)_{\overline{0}}$ is canonically isomorphic to $\mathfrak{sl}(n) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(m)$ and we can choose a homogeneous basis $\{E_A\}_{A=1,\cdots,(n+m)^2-1}$ of $\mathfrak{sl}(n|m)$ such that $E_1, \cdots, E_{\underline{n}^2-1}$ lie

in the isomorphic copy of $\mathfrak{sl}(\underline{n})$ and $E_{\underline{n}^2}, \cdots, E_{n^2+m^2-1}$ in the isomorphic copy of $\mathfrak{gl}(1) \oplus \mathfrak{sl}(\underline{m})$ with $\underline{m} := \min\{n, m\}$. Then the elements $\beta(E_k) := \underline{E}_k, k = 1, \cdots, \underline{n}^2 - 1$, form a basis of $\mathfrak{sl}(\underline{n})$. Denoting the elements of the basis of $\Omega^1(\mathbb{M}(\underline{n}))$ corresponding with $\{\underline{E}_k\}_{k=1,\cdots,\underline{n}^2-1}$ according to (19) and (26) with $\underline{\theta}^k$ the action of the maps $\beta^{(p)}$ can be described alternatively by

$$\begin{aligned} \beta^{(p)}(\omega) &\equiv \beta^{(p)}\left(\sum_{(A_1,\cdots,A_p)\in\mathfrak{I}_p^{n^2+m^2-1|2nm}} \omega_{A_1\cdots A_p} \wedge \theta^{A_1} \wedge \cdots \wedge \theta^{A_p}\right) = \\ &= \sum_{(k_1,\cdots,k_p)\in\mathfrak{I}_p^{\underline{n}^2-1|0}} \beta\left(\omega_{k_1\cdots k_p}\right) \wedge \underline{\theta}^{k_1} \wedge \cdots \wedge \underline{\theta}^{k_p}\,. \end{aligned} \tag{41}$$

If we set $\beta^{(0)} \equiv \beta$ the maps $\beta^{(p)}, p \in \mathbb{N}_0$, extend uniquely to a bihomogeneous, $\mathbb{C}$-linear map $\Omega^g(\mathbb{M}(n|m)) \longrightarrow \Omega(\mathbb{M}(\underline{n}))$ of bidegree $(0, \overline{0})$, which we again denote by β. Because of (41) β is onto and its restriction to $\Omega^g_{\mathcal{Z}^g}(\mathbb{M}(n|m))$ is an surjective homomorphism of bigraded $\mathbb{C}$-algebras onto $\Omega_{\mathcal{Z}}(\mathbb{M}(\underline{n}))$. Furthermore β fulfills

$$\beta \circ L_D = L_{\hat{\beta}(D)} \circ \beta \tag{42}$$

for all $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))_{\overline{0}}$ as well as

$$\beta \circ d = d \circ \beta\,. \tag{43}$$

Consequently β induces a homomorphism $H(\beta) : H(\mathbb{M}(n|m)) \longrightarrow H(\mathbb{M}(\underline{n}))$ of cohomologies in the usual way. Analogous to graded manifold theory [22] this map is an isomorphism.

Proposition 6 *$H(\beta)$ is an isomorphism of bigraded $\mathbb{C}$-algebras, such that both cohomologies $H(\mathbb{M}(n|m))$ and $H(\mathbb{M}(\underline{n}))$ are isomorphic to the Lie algebra cohomology $H(\mathfrak{sl}(\underline{n}); \mathbb{C})$ of $\mathfrak{sl}(\underline{n})$ with trivial coefficients.*

Proof: Using the results of [28] as well as $\mathfrak{Der}^g(\mathbb{M}(n|m)) = \mathfrak{sl}(n|m)^{\mathrm{ad}}$ we find the sequence

$$\begin{aligned} H(\mathbb{M}(n|m)) &\cong H(\mathfrak{sl}(n|m); \mathbb{M}(n|m)) \cong H(\mathfrak{sl}(n|m); \mathbb{C}\, 1_{n+m}) \oplus H(\mathfrak{sl}(n|m); \mathfrak{sl}(n|m)) \cong \\ &\cong H(\mathfrak{sl}(n|m); \mathbb{C}\, 1_{n+m}) \cong H(\mathfrak{sl}(n|m); \mathbb{C}) \end{aligned}$$

of natural isomorphisms between Lie superalgebra cohomologies. In particular we have $H(\mathbb{M}(\underline{n})) \cong H(\mathfrak{sl}(\underline{n}); \mathbb{C})$ (as $\mathbb{N}_0$-graded $\mathbb{C}$-algebra), which is well-known from matrix geometry [5, 6, 9]. Combining the above result with the calculations of the cohomology of $\mathfrak{sl}(n|m)$ with trivial coefficients [11, 12] one can conclude, that $H(\beta)$ is an isomorphism of bigraded $\mathbb{C}$-algebras. □

5 Noncommutative graded symplectic geometry

Generalizing the situation on graded manifolds [22] we call an even, closed graded 2-form $\omega \in \Omega^{g,2}(\mathbb{M}(n|m))$ graded symplectic structure on $\mathbb{M}(n|m)$, if the equation

$$\omega(D, D_M) = DM \tag{44}$$

for all $D \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ possesses a unique solution $D_M \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ for each $M \in \mathbb{M}(n|m)$. The graded vector fields $D_M \in \mathfrak{Der}^g(\mathbb{M}(n|m))$ are called Hamiltonian and the set of all graded Hamiltonian vector fields is denoted by $\mathfrak{Ham}^g(\omega)$.

If $\omega \in \Omega^{g,2}(\mathbb{M}(n|m))$ is graded symplectic structure on $\mathbb{M}(n|m)$ the assignment $M \mapsto D_M$ defines an even $\mathbb{C}$-linear map $D^\omega : \mathbb{M}(n|m) \longrightarrow \mathfrak{Ham}^g(\omega) \subseteq \mathfrak{Der}^g(\mathbb{M}(n|m))$ and one can conclude, that (44) is equivalent to

$$\imath_{D_M}\omega + dM = 0\,. \tag{45}$$

Using (13) we find

$$L_{D_M}\omega = 0 \tag{46}$$

for all $D_M \in \mathfrak{Ham}^g(\omega)$, that is a graded symplectic structure on $\mathbb{M}(n|m)$ is - as usual - invariant with respect to all graded Hamiltonian vector fields.

Via

$$\{M, M'\}_g^\omega := \omega(D_M, D_{M'}) \tag{47}$$

for all $M, M' \in \mathbb{M}(n|m)$ we can introduce a graded Poisson bracket, which has the analogous properties as its graded-commutative pendant.

Proposition 7 $(\mathbb{M}(n|m), \{\cdot,\cdot\}_g^\omega)$ *is a* $\mathbb{C}$*-Lie superalgebra and the graded Poisson bracket fulfills in addition*

$$\begin{aligned} \{M, M'M''\}_g^\omega &= \{M, M'\}_g^\omega M'' + (-1)^{\overline{M}\overline{M}'} M' \{M, M''\}_g^\omega \\ \{1_{n+m}, M\}_g^\omega &= 0 \end{aligned} \tag{48}$$

for all homogeneous $M, M', M'' \in \mathbb{M}(n|m)$*. Moreover, the map* $D^\omega : \mathbb{M}(n|m) \longrightarrow \mathfrak{Ham}^g(\omega)$ *is a homomorphism of Lie superalgebras and*

$$\mathfrak{Ham}^g(\omega) = \mathfrak{Der}^g(\mathbb{M}(n|m))\,. \tag{49}$$

Proof: The properties of $\{\cdot,\cdot\}_g^\omega$ and of D^ω result from the defining properties of the graded symplectic structure ω. From the irreducibility of the adjoint representation of $\mathfrak{sl}(n|m)$ one can deduce $\ker D^\omega = \mathbb{C}\,1_{n+m}$ on the one hand and the injectivity of $D^\omega|_{\mathfrak{sl}(n|m)}$ on the other hand. Then (49) follows because of $\mathfrak{Der}^g(\mathbb{M}(n|m)) = \mathfrak{sl}(n|m)^{\mathrm{ad}}$. □

There exists an essentially unique graded symplectic structure on $\mathbb{M}(n|m)$.

Proposition 8 $d\Theta$ *is a graded symplectic structure on* $\mathbb{M}(n|m)$ *and up to complex multiples it is the only one. The corresponding graded Poisson bracket is given by*

$$\{M, M'\}_g^{d\Theta} = [M, M']_g \tag{50}$$

for all $M, M' \in \mathbb{M}(n|m)$*.*

Proof: The exact, even graded 2-form $c\,d\Theta, c \in \mathbb{C} \setminus \{0\}$ induces via (44) a homomorphism $D^{c\,d\Theta} : \mathbb{M}(n|m) \longrightarrow \mathfrak{Der}^g(\mathbb{M}(n|m))$,

$$D^{c\,d\Theta}(M) = \frac{1}{c}\mathrm{ad}M \tag{51}$$

of Lie superalgebras and the corresponding graded Poisson bracket is given by $\{M, M'\}_g^{c\,d\Theta} = c^{-1}\left[M, M'\right]_g$. The uniqueness property is a consequence of proposition 3, (49) and Schur's Lemma. □

Consequently the extension of the body map β maps a graded symplectic structure ω onto a symplectic structure $\beta(\omega)$. Moreover one has

$$\hat{\beta}\left(D^{\omega}(M)\right) = D^{\beta(\omega)}\left(\beta(M)\right) \tag{52}$$

for all even graded (Hamiltonian) vector fields as well as

$$\beta\left(\{M, M'\}_g^{\omega}\right) = \{\beta(M), \beta(M')\}^{\beta(\omega)} \tag{53}$$

for the graded Poisson bracket of $M, M' \in \mathbb{M}(n|m)_{\overline{0}}$. That is, the relation between $\mathbb{M}(n|m)$ and its body is analogous to the one for graded symplectic manifolds and their respective underlying manifolds.

6 Graded vector bundles over graded matrix algebras

As a synthesis of the definition of graded vector bundles over graded manifolds [1, 17, 22] and the idea how to introduce vector bundles in noncommutative geometry [2, 14] we interpret left, $\mathbb{Z}_2$-graded, finitely generated (graded-projective) $\mathbb{M}(n|m)$-modules as graded vector bundles over $\mathbb{M}(n|m)$ and even $\mathbb{M}(n|m)$-module homomorphisms between such modules as graded vector bundle homomorphisms. We note, that the specifying property of graded projectivity is redundant in the context of left, $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-modules, because on the one hand graded-projective means $\mathbb{Z}_2$-graded plus projective [25] and on the other hand every left $\mathbb{M}(n+m)$-module is projective [26].
Let us denote by $\mathbb{M}(n|m, r|s), r, s \in \mathbb{N}_0, r+s \in \mathbb{N}$, the $\mathbb{C}$-vector space $\mathbb{M}(n+m, r+s)$ together with the $\mathbb{Z}_2$-grading defined by

$$\begin{aligned}
\mathbb{M}(n|m, r|s)_{\overline{0}} &:= \left\{ v = \begin{pmatrix} v_1 & 0 \\ 0 & v_4 \end{pmatrix} \Big| \; v_1 \in \mathbb{M}(n, r), v_4 \in \mathbb{M}(m, s) \right\} \\
\mathbb{M}(n|m, r|s)_{\overline{1}} &:= \left\{ v = \begin{pmatrix} 0 & v_2 \\ v_3 & 0 \end{pmatrix} \Big| \; v_2 \in \mathbb{M}(n, s), v_3 \in \mathbb{M}(m, r) \right\}.
\end{aligned} \tag{54}$$

With respect to ordinary matrix multiplication $\mathbb{M}(n|m, r|s)$ becomes a left, $\mathbb{Z}_2$-graded, finitely generated $\mathbb{M}(n|m)$-module and these examples constitute essentially all graded vector bundles over $\mathbb{M}(n|m)$.

Proposition 9 *If $\mathcal{V}$ is a graded vector bundle over $\mathbb{M}(n|m)$ then there exist unique numbers $r, s \in \mathbb{N}_0, r+s \in \mathbb{N}$ and a graded vector bundle isomorphism $\phi : \mathcal{V} \longrightarrow \mathbb{M}(n|m, r|s)$. $\mathcal{V}$ is graded-free if and only if there are natural numbers $p, q \in \mathbb{N}_0, p+q \in \mathbb{N}$, such that*

$$\begin{aligned} pn + qm &= r \\ pm + qn &= s\,. \end{aligned} \tag{55}$$

Proof: The existence of the isomorphisms are implied by the graded simplicity of $\mathbb{M}(n|m, 1|0)$ and $\mathbb{M}(n|m, 0|1)$ and the fact, that every left, $\mathbb{Z}_2$-graded, finitely generated $\mathbb{M}(n|m)$-module is the homomorphic image of a left, $\mathbb{Z}_2$-graded, graded-free $\mathbb{M}(n|m)$-module with homogeneous basis of suitable cardinality $p|q$. Because all $\mathbb{M}(n|m)$-module isomorphisms are $\mathbb{C}$-vector space isomorphisms in particular, the uniqueness statement and (55) follow. □

After this "miniature-classification" we develop graded differential geometry on a fixed graded vector bundle $\mathcal{V}$ generalizing the treatment of noncommutative geometry [2, 5, 14, 24] on the one hand and the one of supergeometry [1] on the other hand.
So we first define the set $\Omega^g(\mathcal{V})$ of $\mathcal{V}$-valued graded forms according to

$$\Omega^g(\mathcal{V}) \equiv \bigoplus_{p\in\mathbb{N}_0} \Omega^{g,p}(\mathcal{V}) := \Omega^g(\mathbb{M}(n|m))\hat{\otimes}_{\mathbb{M}(n|m)}\mathcal{V}\,. \tag{56}$$

$\Omega^g(\mathcal{V})$ is a left $\mathbb{N}_0\times\mathbb{Z}_2$-bigraded $\Omega^g(\mathbb{M}(n|m))$-module in a natural way and each $\Omega^{g,p}(\mathcal{V}), p \in \mathbb{N}_0$, is a left, $\mathbb{Z}_2$-graded, finitely generated $\mathbb{M}(n|m)$-module in particular. The product will again be denoted by $\wedge$.
A connection on $\mathcal{V}$ is an even $\mathbb{C}$-linear map $\nabla : \mathcal{V} \longrightarrow \Omega^{g,1}(\mathcal{V})$ such that

$$\nabla\,(Mv) = dM \otimes v + M \wedge \nabla v \tag{57}$$

is fulfilled for all $M \in \mathbb{M}(n|m), v \in \mathcal{V}$. Connections always exist due to (graded) projectivity.

Proposition 10 *Let $\mathcal{V}$ be a graded vector bundle over $\mathbb{M}(n|m)$. Then there exists a graded-free vector bundle $\mathcal{V}^{p|q}$ over $\mathbb{M}(n|m)$ with homogeneous basis $\{\ \varepsilon_A \mid \varepsilon_A \in \mathcal{V}^{p|q}_{\bar{0}}, A = 1, \cdots, p; \varepsilon_A \in \mathcal{V}^{p|q}_{\bar{1}}, A = p{+}1, \cdots, p{+}q\}, p, q \in \mathbb{N}_0, p{+}q \in \mathbb{N}$, together with an even, idempotent endomorphism $P : \mathcal{V}^{p|q} \longrightarrow \mathcal{V}^{p|q}$ and an isomorphism $\varphi : \mathcal{V} \longrightarrow \mathrm{im}P$ of $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-modules. The map $\nabla_d : \mathcal{V}^{p|q} \longrightarrow \Omega^{g,1}(\mathcal{V}^{p|q})$ defined by*

$$\nabla_d(v) \equiv \nabla_d\left(\sum_{A=1}^{p+q} v^A\varepsilon_A\right) := \sum_{A=1}^{p+q} dv^A \otimes \varepsilon_A \tag{58}$$

is a connection on $\mathcal{V}^{p|q}$ and

$$\nabla_{Pd} := \mathrm{Id}_{\Omega^{g,1}(\mathbb{M}(n|m))} \otimes \varphi^{-1} \circ \mathrm{Id}_{\Omega^{g,1}(\mathbb{M}(n|m))} \otimes P \circ \nabla_d \circ \varphi \tag{59}$$

is a connection on $\mathcal{V}$. A map $\nabla : \mathcal{V} \longrightarrow \Omega^{g,1}(\mathcal{V})$ is a connection on $\mathcal{V}$ if and only if it is of the form

$$\nabla = \nabla_{Pd} + \alpha\,, \tag{60}$$

where $\alpha : \mathcal{V} \longrightarrow \Omega^{g,1}(\mathcal{V})$ is an even homomorphism of $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-modules.

Proof: Analogous to the ungraded case [2, 14]. □

Quite generally we will denote the $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-bimodule of graded $(p+q)\times(p+q)$-matrices over a $\mathbb{Z}_2$-graded bimodule $\mathcal{B}$ with $\mathbb{M}(p|q;\mathcal{B})$. It is a $\mathbb{M}(n|m)$-bimodule in a natural way and $\mathbb{Z}_2$-graded by declaring those matrices with even diagonal entries and odd off-diagonal entries as even and those with odd diagonal entries and even off-diagonal entries as odd. Adopting the notation of the above proposition we introduce homogeneous generators

$$\eta_A := \varphi^{-1} \circ P(\varepsilon_A) \tag{61}$$

of $\mathcal{V}$ as well as an even matrix $(P_A^B) \in \mathbb{M}(p|q;\mathbb{M}(n|m))_{\overline{0}}$ via

$$P(\varepsilon_A) =: \sum_{B=1}^{p+q} P_A^B \varepsilon_B\,. \tag{62}$$

Then

$$(\nabla - \nabla_{Pd})\,(\eta_A) = \alpha(\eta_A) =: \sum_{B=1}^{p+q} \alpha_A^B \otimes \eta_B \tag{63}$$

establishes a bijective correspondence between the set of all connections on $\mathcal{V}$ and the set $P\mathbb{M}(p|q;\Omega^{g,1}(\mathbb{M}(n|m)))_{\overline{0}}P$, which consists of those $(\alpha_A^B) \in \mathbb{M}(p|q;\Omega^{g,1}(\mathbb{M}(n|m)))_{\overline{0}}$ fulfilling $\alpha_A^B = \sum_{C,D=1}^{p+q} P_A^C \wedge \alpha_C^D \wedge P_D^B$ (for $\mathcal{V} = \mathcal{V}^{p|q}$ set $P = \varphi = \mathrm{Id}_{\mathcal{V}^{p|q}}$). The graded 1-forms α_A^B are called connection forms of the connection $\nabla = \nabla_{Pd} + \alpha$.
If ∇ is a connection on a graded vector bundle $\mathcal{V}$ we can introduce a $\mathbb{C}$-linear map $\Omega^g(\mathcal{V}) \longrightarrow \Omega^g(\mathcal{V})$, again denoted by ∇, via

$$\nabla\,(\omega \otimes v) = d\omega \otimes v + (-1)^p \omega \wedge \nabla v \tag{64}$$

for all $v \in \mathcal{V}, \omega \in \Omega^{g,p}(\mathbb{M}(n|m)), p \in \mathbb{N}_0$. This homogeneous map of bidegree $(1,\overline{0})$ extends the original connection if we identify $\mathcal{V}$ with $\Omega^{g,0}(\mathcal{V})$. Moreover it fulfills

$$\nabla\,(\omega' \wedge \omega \otimes v) = d\omega' \wedge (\omega \otimes v) + (-1)^{p'} \omega' \wedge \nabla(\omega \otimes v) \tag{65}$$

for all $v \in \mathcal{V}, \omega \in \Omega^{g,p}(\mathbb{M}(n|m)), \omega' \in \Omega^{g,p'}(\mathbb{M}(n|m)), p, p' \in \mathbb{N}_0$, and this property determines the extension of the connection uniquely.
The curvature of a connection ∇ on a graded vector bundle $\mathcal{V}$ is defined as

$$\nabla^2 \equiv \nabla \circ \nabla : \mathcal{V} \longrightarrow \Omega^{g,2}(\mathcal{V})\,. \tag{66}$$

It is an even homomorphism of $\mathbb{Z}_2$-graded $\mathbb{M}(n|m)$-modules and one can describe its action on an element $v = \sum_{A=1}^{p+q} \varphi(v)^A \eta_A \in \mathcal{V}$ according to

$$\nabla^2(v) =: \sum_{A,B=1}^{p+q} \varphi(v)^A \wedge R_A^B \otimes \eta_B \tag{67}$$

with a uniquely determined matrix $(R_A^B) \in P\mathbb{M}(p|q;\Omega^{g,2}(\mathbb{M}(n|m)))_{\overline{0}}P$. The graded 2-forms R_A^B are called curvature forms and they can be expressed according to

$$R_A^B = -\sum_{C=1}^{p+q} \alpha_A^C \wedge \alpha_C^B + \sum_{C,D=1}^{p+q} \left(P_A^C \wedge d\alpha_C^D \wedge P_D^B - P_A^C \wedge dP_C^D \wedge dP_D^B \right) \tag{68}$$

in terms of the connection forms α_A^B of the connection. Moreover they have to fulfill the Bianchi identity

$$\sum_{C,D=1}^{p+q} P_A^C \wedge dR_C^D \wedge P_D^B - \sum_{C=1}^{p+q} \left(\alpha_A^C \wedge R_C^B - R_A^C \wedge \alpha_C^B \right) = 0\,. \tag{69}$$

Let us finally analyze the space of flat connections, that is the set of all connections with vanishing curvature. We will not do this in complete generality but only for a graded-free vector bundle $\mathcal{V}^{1|0}$ with an even basis element ε.

Proposition 11 *A connection on $\mathcal{V}^{1|0}$ is flat if and only if its connection form $\alpha \in \Omega^{g,1}(\mathbb{M}(n|m))_{\overline{0}}$ is either given by*

$$\alpha = \Theta \tag{70}$$

or by

$$\alpha = \Theta - \sum_{A=1}^{(n+m)^2-1} f(E_A) \wedge \theta^A\,, \tag{71}$$

where $\{E_A\}$ is the homogeneous basis of $\mathfrak{sl}(n|m)$ "corresponding" with $\{\theta^A\}$ and f is some automorphism of $\mathfrak{sl}(n|m)$.

Proof: Let us introduce an even graded 1-form $\rho = \sum_{A=1}^{(n+m)^2-1} \rho_A \wedge \theta^A$ according to $\alpha =: \Theta - \rho$. Using proposition 4 we find, that the curvature form is given by

$$R = \frac{1}{2} \sum_{A,B=1}^{(n+m)^2-1} \Omega_{AB} \wedge \theta^A \wedge \theta^B \tag{72}$$

with

$$\Omega_{AB} = [\rho_B, \rho_A]_g - \sum_{C=1}^{(n+m)^2-1} c_{BA}^C \rho_C\,. \tag{73}$$

Because the vanishing of the curvature is equivalent to $\Omega_{AB} = 0$ the proposition follows from the simplicity of $\mathfrak{sl}(n|m)$. □

That is, we have the same situation as in ordinary matrix geometry [5, 6]: There exist different "classes" of flat connections. Here "class" refers to the action of the group of automorphisms of the graded vector bundle on the space of connections, which can be introduced analogous to the ungraded case. The connection ∇_d and the one associated with the connection form Θ will lie in different classes, because the latter is invariant. However, if one does not restrict the space of connections by a suitable compatibility requirement with respect to a graded hermitian structure there will exist even more than two classes of flat connections.

7 Concluding remarks

We have developed the graded differential geometry of graded matrix algebras and shown that the results of matrix geometry [5, 6] carry over to the $\mathbb{Z}_2$-graded setting. In addition we found

a natural noncommutative analogue of the body map, which allows us to view graded matrix geometries as true noncommutative generalizations of graded manifolds.
Whereas in ordinary differential geometry one integrates forms this is not true in supergeometry. Except from the before mentioned body map, which plays a central role in the global theory of Berezin integration [18], we completely excluded the integral geometry of graded matrix algebras. We plan to treat this together with metric aspects in a separate work.
Beside its immediate application for the construction of (graded) differential calculi on fuzzy (super)manifolds [24, 16] the developments of this article offer another perspective. The extension of space-time by matrix geometries led to interesting new gauge models. In particular the existence of different gauge orbits of flat connections in matrix geometry is the origin of the appearance of the Higgs effect [5, 7]. The possibility of extending the structures of matrix geometry to $\mathbb{Z}_2$-graded matrix algebras suggests to think about similar "supersymmetric" noncommutative extensions of space-time.

A Associative product of supertrace-free, graded matrices

Let $\{E_A | E_A \in \mathfrak{sl}(n|m)_{\overline{0}}, A = 1, \cdots, n^2+m^2-1; E_A \in \mathfrak{sl}(n|m)_{\overline{1}}, A = n^2+m^2, \cdots, (n+m)^2-1\}$ be a homogeneous basis of $\mathfrak{sl}(n|m), n \neq m$. Our aim of this appendix is to investigate the associative product of the homogeneous matrices E_A in a similar way as it was done in [23] for trace-free, hermitian matrices.

If we introduce a graded anticommutator

$$[M, M']_g^+ := MM' + (-1)^{\overline{M}\overline{M}'} M'M \tag{A1}$$

of two homogeneous $M, M' \in \mathbb{M}(n|m)$ we find the relations

$$\begin{aligned} \Big[M, [M', M'']_g\Big]_g - \Big[M, [M', M'']_g^+\Big]_g^+ + (-1)^{\overline{M}'\overline{M}''} \Big[[M, M'']_g^+, M'\Big]_g^+ &= 0 \\ \Big[[M, M']_g^+, M''\Big]_g - \Big[M, [M', M'']_g\Big]_g^+ - (-1)^{\overline{M}'\overline{M}''} \Big[[M, M'']_g, M'\Big]_g^+ &= 0 \end{aligned} \tag{A2}$$

between the graded commutator and the graded anticommutator of homogeneous $M, M', M'' \in \mathbb{M}(n|m)$.
Because $\{E_A, 1_{n+m}\}_{A=1,\cdots,(n+m)^2-1}$ forms a homogeneous basis of $\mathbb{M}(n|m)$ the graded anticommutator of E_A and E_B can be written according to

$$[E_A, E_B]_g^+ = \sum_{C=1}^{(n+m)^2-1} d_{AB}^C E_C + g_{AB} 1_{n+m} \tag{A3}$$

with uniquely determined coefficients $d_{AB}^C, g_{AB} \in \mathbb{C}$. Then the associative product of E_A and E_B is given by

$$E_A E_B = \frac{1}{2} \sum_{C=1}^{(n+m)^2-1} (c_{AB}^C + d_{AB}^C) E_C + \frac{1}{2} g_{AB} 1_{n+m}\,. \tag{A4}$$

Independent of the specific choice of the homogeneous basis $\{E_A\}_{A=1,\cdots,(n+m)^2-1}$ there exist a lot of relations between the "structure constants" c_{AB}^C, d_{AB}^C and g_{AB} which we summarize in

Proposition A

(i.) *c_{AB}^{C} and d_{AB}^{C} vanish if $\overline{E}_A + \overline{E}_B + \overline{E}_C = \overline{1}$ and g_{AB} vanishes if $\overline{E}_A + \overline{E}_B = \overline{1}$.*

(ii.)

$$\begin{aligned} \sum_{B=1}^{(n+m)^2-1} (-1)^{\overline{E}_B} c_{AB}^{B} &= 0 \\ \sum_{B=1}^{(n+m)^2-1} (-1)^{\overline{E}_B} d_{AB}^{B} &= 0 \end{aligned} \tag{A5}$$

(iii.) *c_{ABC} and d_{ABC}, defined via*

$$\begin{aligned} c_{ABC} &:= \sum_{D=1}^{(n+m)^2-1} c_{AB}^{D} g_{DC} \\ d_{ABC} &:= \sum_{D=1}^{(n+m)^2-1} d_{AB}^{D} g_{DC}\,, \end{aligned} \tag{A6}$$

are totally antisymmetric respectively totally symmetric in the $\mathbb{Z}_2$-graded sense.

(iv.)

$$\sum_{E=1}^{(n+m)^2-1} \left\{(-1)^{\overline{E}_A\overline{E}_C} c_{BC}^{E} c_{AE}^{D} + (-1)^{\overline{E}_B\overline{E}_A} c_{CA}^{E} c_{BE}^{D} + (-1)^{\overline{E}_C\overline{E}_B} c_{AB}^{E} c_{CE}^{D}\right\} = 0$$

$$\begin{aligned} \sum_{E=1}^{(n+m)^2-1} \left\{c_{BC}^{E} c_{AE}^{D} - d_{AB}^{E} d_{EC}^{D} + (-1)^{\overline{E}_A\overline{E}_C + \overline{E}_B\overline{E}_C} d_{CA}^{E} d_{EB}^{D}\right\} + \\ +2(-1)^{\overline{E}_A\overline{E}_C + \overline{E}_B\overline{E}_C} g_{CA}\delta_B^D - 2g_{AB}\delta_C^D = 0 \end{aligned} \tag{A7}$$

$$\sum_{E=1}^{(n+m)^2-1} \left\{d_{AB}^{E} c_{EC}^{D} - c_{BC}^{E} d_{AE}^{D} - (-1)^{\overline{E}_B\overline{E}_C} c_{AC}^{E} d_{EB}^{D}\right\} = 0$$

(v.) *If $K_{AB} := K(E_A, E_B)$, where K is the Killing form of $\mathfrak{sl}(n|m)$, then*

$$\begin{aligned} K_{AB} = (n-m)^2 g_{AB} &= \sum_{C,D=1}^{(n+m)^2-1} (-1)^{\overline{E}_C} c_{AD}^{C} c_{BC}^{D} = \\ &= \frac{(n-m)^2}{(n-m)^2-4} \sum_{C,D=1}^{(n+m)^2-1} (-1)^{\overline{E}_C} d_{AD}^{C} d_{BC}^{D}\,. \end{aligned} \tag{A8}$$

(vi.) *Denoting by g^{AB} the components of the matrix inverse to (g_{AB}), then*

$$\begin{aligned} \sum_{B,C=1}^{(n+m)^2-1} g^{BC} c_{BC}^{A} &= 0 \\ \sum_{B,C=1}^{(n+m)^2-1} g^{BC} d_{BC}^{A} &= 0\,. \end{aligned} \tag{A9}$$

(vii.)

$$\sum_{C,D,E=1}^{(n+m)^2-1} g^{CD} c_{CE}^{A} c_{DB}^{E} = (n-m)^2 \delta_B^A$$

$$\begin{aligned}
\sum_{C,D,E=1}^{(n+m)^2-1} g^{CD} c_{CE}^{A} d_{DB}^{E} &= 0 \\
\sum_{C,D,E=1}^{(n+m)^2-1} g^{CD} d_{CE}^{A} d_{DB}^{E} &= \Big((n-m)^2-4\Big)\,\delta_B^A
\end{aligned} \tag{A10}$$

(viii.)

$$\begin{aligned}
\sum_{D,E,F,G=1}^{(n+m)^2-1} (-1)^{\overline{E}_A\overline{E}_E} g^{DE} c_{EB}^{F} c_{AF}^{G} c_{DG}^{C} &= \frac{1}{2}(n-m)^2 c_{AB}^{C} \\
\sum_{D,E,F,G=1}^{(n+m)^2-1} (-1)^{\overline{E}_A\overline{E}_E} g^{DE} c_{EB}^{F} c_{AF}^{G} d_{DG}^{C} &= -\frac{1}{2}(n-m)^2 d_{AB}^{C} \\
\sum_{D,E,F,G=1}^{(n+m)^2-1} (-1)^{\overline{E}_A\overline{E}_E} g^{DE} c_{EB}^{F} d_{AF}^{G} d_{DG}^{C} &= -\frac{1}{2}\Big((n-m)^2-4\Big)\,c_{AB}^{C} \\
\sum_{D,E,F,G=1}^{(n+m)^2-1} (-1)^{\overline{E}_A\overline{E}_E} g^{DE} d_{EB}^{F} d_{AF}^{G} d_{DG}^{C} &= \frac{1}{2}\Big((n-m)^2-12\Big)\,d_{AB}^{C}
\end{aligned} \tag{A11}$$

Proof: (i.) is a reformulation of the homogeneity of $\{E_A\}$. The first line of (A8) as well as (iii.) result from $K_{AB} = 2(n-m)\mathrm{Tr}_s(E_A E_B)$. (ii.) is a consequence of $\mathrm{Tr}_s(\mathrm{ad}E_A) = 0$ and of $\sum_{B,C} \mathrm{Tr}_s(g^{BC}\,[E_B, E_C]_g^+\,E_A) = 0$. (iv.) is a reformulation of the graded Jacobi identity and (A2). Using the second equation (A7) one deduces the second line of (A8). (vi.) follows from (ii.) and (iii.). The left hand side of the first equation (A10) is essentially the second-order Casimir operator of $\mathfrak{sl}(n|m)$ in the adjoint representation. The second part of (A10) follows from (iii.), whereas the third equation is a consequence of the first part together with (iv.) and (vi.). The relations (viii.) are results of calculations using (iii.),(iv.),(vi.) and (vii.). $\square$

Acknowledgment

The authors would like to thank W. Bulla, J. Madore, P. Prešnajder and L. Pittner for helpful discussions and the "Fonds zur Förderung der wissenschaftlichen Forschung (FWF)" for support funding.

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